ELASTIC STABILTY ANALYSIS OF RECTANGULAR PLATE WITH TWO OPPOSITE EDGES CLAMPED AND OTHER EDGES FREE USING POLYNOMIAL DISPLACEMENT FUNCTIONS

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ABSTRACT

This paper presents elastic stability analysis of rectangular plate with two opposite edges clamped and other edges free (CFCF) using polynomial displacement function derived from the Taylor-Maclaurin power series expansion. The geometric configurations of the plate were approximated and substituted in an indirect variati on equation formulated by the Galerkin method to obtain the critical buckling load equation. Results obtained at aspect ratios ranging from 0.1 to 1.0 at 0.1 interval showed correlation coefficients of 0.9995 and 0.9997 when K-values for CFCF bounded plate were compared to K-values of previous researches on plate with all edges clamped (CCCC). These values reflected no statistical difference when tested and indicate that the CFCF bounded rectangular plate could be an adequate substitution for the rectangular plate clamped at all edges (CCCC) that it has been compared against. Also, based on the rapid convergence of the polynomial displacement function adopted, the approach presented in this study gives a good approximation to the exact solution in analyzing the stability of clamped rectangular plates with opposite edges (in the x-direction) free of supports.

Keywords: Approximate solution, Critical buckling load, free edge support condition, Indirect variational, Polynomial displacement function

a	Length of the primary dimension of the plate
b	length of the secondary dimension of the plate
CCCC	Rectangular plate with all edges clamped
CFCF	Rectangular plate with opposite edge clamped (η-axis) and the other edges free
D	Flexural rigidity of a plate
E	Modulus of elasticity in compression and tension
K	Buckling coefficient
m	The number of terms in the truncated series with respect to the x direction
n	The number of terms in the truncated series with respect to the y direction
N_x,N_y	In-plane Normal forces per unit length of section of a plate perpendicular to the x- and y-axis respectively
$N_{x(cr)}$	Critical buckling load (force)

$N_{xy,} N_{yx}$ In-plane shearing forces per unit length of section of plate acting in the XY and YX plane respectively
P Aspect ratio
t Thickness of plate
w Plate displacement in the z direction in function of x and y
x, y, z Rectangular coordinates
Partial derivative of a function
ξ Non dimensional axis (quantity) parallel to x axis $\xi = x/a$
η Non dimensional axis (quantity) parallel to y axis $\eta = y/b$
π Pie (22/7)

INTRODUCTION

Plates, characteristically flat, can result in buckling or structural instability by the transition from stability to instability as a response of the in-plane reaction of the middle plane when subjected to normal compressive forces or shearing forces (Szilard, 2004). This response as observed by Gallagher (1971) can trigger general buckling of a larger structure because of a redistribution of loads and can also affect the structure sufficiently enough to cause failure from excessive displacement or fatigue or aero-elastic phenomena

Rectangular plates as found in structures as bridge decks, jetty platforms and even diving towers are demonstrations of the application of plates with opposite edges free of supports, and also a representation of the extensive use of plates applicable in all fields of engineering based on the distinct advantages inherent in plates (Ventsel and Krauthammer, 2001). Timoshenko and Woinowsky-Krieger (1959) presented plates with free edges as condition that eliminates bending moments, twisting moments and shear forces along such edges. Tajdari, *et al*. (2011) studied the effects of plate-support condition on buckling strength of rectangular perforated plates under linearly varying in-plane normal load and concluded that for a plate subjected to compressive loading, free edge conditions enhanced the mechanical buckling strength of perforated plates more effectively than the fixed edge conditions and clamped edges offered more buckling resistance strength compared to the simply supported edges.

Reddy (1999) suggested equilibrium or Euler, energy and dynamic methods as approaches to the solution of elastic and inelastic stability problems and used by Timoshenko and Woinowsky-Krieger (1959) by the application of Levy's and Navier's trigonometric series to obtain deflection equations for simply supported edges only. Iyengar (1988) implementation of the variation of total potential energy method assuming double trigonometric displacement equations in obtaining buckling critical load $N_{x(cr)}$ in first mode for various support conditions as rectangular plates with all edges clamped (CCCC), rectangular plates with opposite edges clamped and the other edges simply supported (CSCS) and rectangular plates with all edges simply supported (SSSS), has displayed the energy method as a veritable alternative to buckling problems of structural plates. Bhaskar (2007) presented the numerical approach as an efficient method in obtaining approximate solutions to instability problems but more recent studies have revealed the formulation of polynomial displacement function from the Taylor-Maclaurin power series expansion as a simpler approach to obtaining approximate solutions with close convergence to the exact solutions.

Ibearugbulem (2012) applied these polynomial series to the Ritz variational method to obtain critical buckling load equation for rectangular plate with adjacent edges clamped and the other edges simply supported (CCSS) and Okpara (2014) studied the application of variational method in stability analysis of aircraft structures highlighted the critical buckling load $N_{x(cr)}$ for various panels in the study viz: CCCC, CCSS, and CSCS amongst others cases considered. Solution has also been sought for plate in the inelastic region by Eziefula, *et al*. (2013) in adopting the series in the inelastic buckling analysis of axially compressed thin CCCC plates using the Taylor-Maclaurin displacement function to obtain inelastic buckling load for the all edges clamped plate (CCCC). Research works on rectangular plates with free edge has also been conducted as shown by Ezeh, *et al*. (2014) with the use of polynomial shape function $W(R,Q) = (R - 2R^3 + R^4)(8Q - 4Q^3 + Q^4)$ in obtaining the stability analysis equation for an isotropic SSFS rectangular plates adopting the Ritz variational energy method; Ebirem, *et al.* (2014) obtained new energy function for fundamental natural frequencies with respect to aspect ratios, the research was based on Ritz's total potential energy and Taylor series deflection equation, formulated for CSCF and SCFC panels. Okpiabhele (2015) predicted the stability of slab bridge deck by the indirect variation approach using MATLAB and obtained critical buckling load and stress values for boundary conditions as CFCF, rectangular plates with one edge clamped, its opposite edge simply supported and the other two opposite edges Free (CFSF), and rectangular plates with opposite edges simply supported and the other two edges free (SFSF) which are found valuable in the analysis carried out on this paper.

This paper investigates the need of an adequate replacement of rectangular plates with all edges supported by rectangular plates with opposite edges (in the x-axis) free of supports. A comprehensive buckling analysis approach is presented for the numerical solution of the thin rectangular plate subjected to compressive loads in the uniaxial direction, and this provides the basis for comparison with previous studies.

DERIVATION OF DISPLACEMENT FUNCTION (CHARACTERISTIC ORTHOGONAL POLYNOMIAL) FOR RECTANGULAR PLATE WITH CFCF EDGES CONDITIONS

Ibearugbulem (2012) formulated a general shape function for rectangular plates from the Taylor-Maclaurin's series, also referred to as Characteristic Orthogonal Polynomial (COP) displacement equation represented in the form:

$$w = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} X_m x^m Y_n y^n \quad (1)$$

Replacing the variables x and y with variables in the non-dimensional form,

$X = a\xi$; $Y = b\eta$ and $P = \frac{a}{b}$, then

$$w = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m b_n \xi^m \eta^n \quad (2)$$

Since the buckling equation of plate, is a fourth order differential and the density of the plate is constant, then, the value of m and n in equation (2) has to be equal to 4. For other variation of loading, higher value of the power m and n may be applied (Onyeyili, 2012).
Therefore, substituting m = 4 and n = 4 in equation (2), gives:

$$w(\xi,\eta) = \sum_{m=0}^{4}\sum_{n=0}^{4} a_m b_n \xi^m \eta^n \quad (3)$$

Expanding equation (3)

$$w(\xi,\eta) = (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4)(b_0 + b_1\eta + b_2\eta^2 + b_3\eta^3 + b_4\eta^4) \quad (4)$$

For a rectangular plate with two opposite edges clamped (η-axis) and the other two edges free (CFCF)

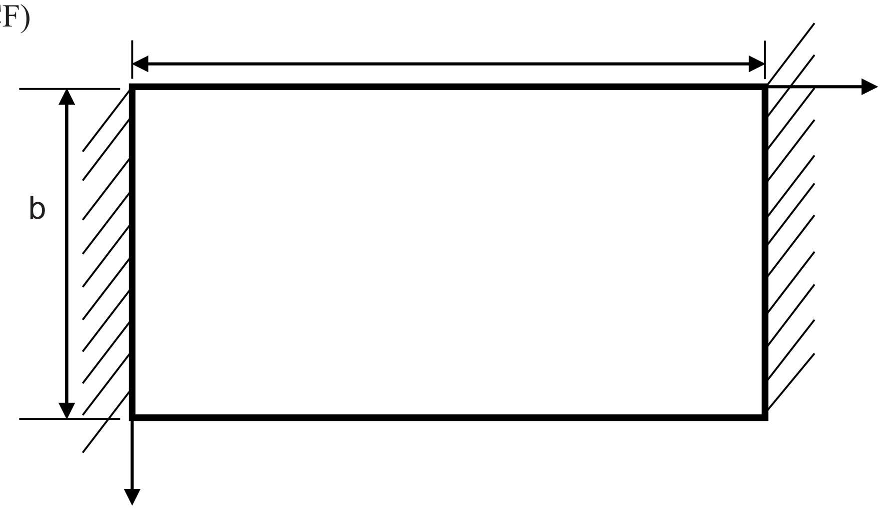

Figure 1: Schematic diagram of a slab bridge deck with CFCF support condition
The edge conditions are:

$$w_{(\xi = 0)} = \frac{\partial w}{\partial x}_{(\xi = 0)} = 0 \quad (5)$$

$$w_{(\xi = 1)} = \frac{\partial w}{\partial x}_{(\xi = 1)} = 0 \quad (6)$$

$$\frac{\partial^2 w}{\partial y^2}_{(\eta = 0)} = \frac{\partial^3 w}{\partial y^3} + (2 - \vartheta)\frac{\partial^3 w}{\partial y \partial x^2}_{(\eta = 0)} = 0 \quad (7)$$

$$\frac{\partial^2 w}{\partial y^2}_{(\eta=1)} = \frac{\partial^3 w}{\partial y^3} + (2-\vartheta)\frac{\partial^3 w}{\partial y \partial x^2}_{(\eta=1)} = 0 \qquad (8)$$

Substituting $w(\xi,\eta) = (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4)(b_0 + b_1\eta + b_2\eta^2 + b_3\eta^3 + b_4\eta^4)$ and its partial derivatives into the support equations (5) to (8) in the $A = a_4\ b_4$ yields the displacement function for CFCF plates as yields the displacement function for CFCF plates as

$$w = A(\xi^2 - 2\xi^3 + \xi^4)(\eta - 2\eta^3 + \eta^4) \qquad (9)$$

INDIRECT VARIATIONAL APPROACH TO ELASTIC BUCKLING GOVERNING EQUATION

Galerkin method was found useful as demonstrated by Ventsel and Krauthemmer (2001) in the solution of boundary value problems of solid mechanics and elasticity and belongs to the indirect method of analysis. An approximate solution of the differential equation in 2D domain (Ω) is represented as:

$$L[w(x,y)]_{\Omega} = P\ (x,y) \qquad (10)$$

where w(x,y) s an unknown function of 2 variables which can be extended to 3D problems as well, P(x,y) is the given load term defined also in the domain $L[...] = D\nabla^2\nabla^2(...)$

Equation (10) can also be re-written, when all variable is taken to a side of the equation and summed for various small units considered as

$$\iint [L\,(w) - P] f_i\,(x,y) dxdy = 0 \qquad (11)$$

When applied to solve rectangular plate problems give:

$$L\,[w(x,y)\,] \equiv D\left[\nabla^2\,\nabla^2\,\left(w(x,y)\right)\right] = N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y\frac{\partial^2 w}{\partial y^2} \qquad (12)$$

For a rectangular plate that is uniaxially loaded, i.e. in the x direction, Nxy = Ny = 0

$$D\left[\nabla^2\,\nabla^2\,\left(w(x,y)\right)\right] = N_x\frac{\partial^2 w}{\partial x^2} \qquad (13)$$

This can be re-arranged as

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - N_x\frac{\partial^2 w}{\partial x^2} = 0 \qquad (14)$$

Substituting equation (14) into (11) gives,

$$\iint \left[\mathrm{D} \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) - N_x \frac{\partial^2 w}{\partial x^2} \right] \mathrm{f_i}\ (\mathrm{x,y}) \mathrm{dxdy} = 0 \quad (15)$$

Expanding and taking $\frac{\partial^2 w}{\partial x^2}$ as $\frac{\partial}{\partial \mathrm{x}} \left(\frac{\partial \mathrm{w}}{\partial \mathrm{x}} \right)$. Then N_x is made the subject of the formula to solve for the critical uni-axial compressive force,

$$\iint \mathrm{D} \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) \mathrm{f_i}\ (\mathrm{x,y}) \mathrm{dxdy} = \iint N_x \frac{\partial}{\partial \mathrm{x}} \left(\frac{\partial \mathrm{w}}{\partial \mathrm{x}} \right) \mathrm{f_i}\ (\mathrm{x,y}) \mathrm{dxdy} \quad (16)$$

$$N_x = \frac{\iint \mathrm{D} \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) \mathrm{f_i}(\mathrm{x,y}) \mathrm{dxdy}}{\iint \frac{\partial}{\partial \mathrm{x}} \left(\frac{\partial w}{\partial \mathrm{x}} \right) \mathrm{f_i}(\mathrm{x,y}) \mathrm{dxdy}} \quad (17)$$

In the non-dimensional form, $\mathrm{X} = \mathrm{a}\xi$ and $\mathrm{Y} = \mathrm{b}\eta$. N_x can be put as:

$$N_x = \frac{\iint \mathrm{D} \left(\frac{\partial^4 w}{a^4 \partial \xi^4} + 2 \frac{\partial^4 w}{a^2 b^2 \partial \xi^2 \partial \eta^2} + \frac{\partial^4 w}{b^4 \partial \eta^4} \right) \mathrm{f_i}(\xi,\eta) \mathrm{abd}\xi \mathrm{d}\eta}{\iint \frac{\partial}{\mathrm{a}^4 \partial \xi} \left(\frac{\partial w}{\partial \xi} \right) \mathrm{f_i}(\xi,\eta) \mathrm{abd}\xi \mathrm{d}\eta} \quad (18)$$

Dividing through by b^4 and substituting $P = \frac{a}{b}$ into the equation (18) yields;

$$N_x = \frac{\iint \frac{\mathrm{D}}{\mathrm{b}^2} \left(\frac{\partial^4 w}{P^3 \partial \xi^4} + 2 \frac{\partial^4 w}{P \partial \xi^2 \partial \eta^2} + \mathrm{P} \frac{\partial^4 w}{\partial \eta^4} \right) \mathrm{f_i}(\xi,\eta) \mathrm{d}\xi \mathrm{d}\eta}{\frac{1}{\mathrm{P}} \iint \frac{\partial}{\partial \xi} \left(\frac{\partial w}{\partial \xi} \right) \mathrm{f_i}(\xi,\eta) \mathrm{d}\xi \mathrm{d}\eta} \quad (19)$$

Equation (19) is the Indirect Variational Load Function for a buckled rectangular plate along the edges perpendicular to the x direction in a non-dimensional form.

FORMULATION OF STABILITY EQUATION BY THE INDIRECT VARIATIONAL APPROACH FOR RECTANGULAR PLATE WITH CFCF EDGES CONDITIONS

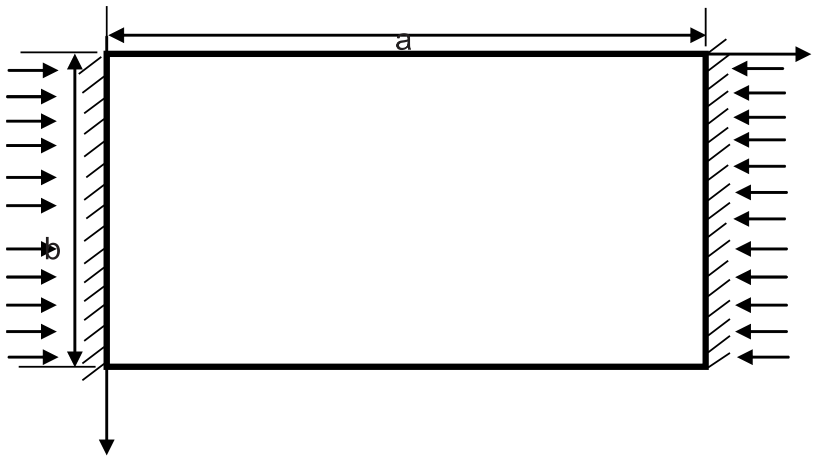

Figure 2: Schematic diagram of rectangular plate under uniaxial in-plane compression

The displacement function for rectangular plate with CFCF edge conditions (9) is:

$w = A(\xi^2 - 2\xi^3 + \xi^4)(\eta - 2\eta^3 + \eta^4)$, *where* $A = a_4 b_4$.

Substituting (9) into (19) yields;

$$\frac{D}{P^3 b^2}\int_0^1\int_0^1 \frac{\partial^4 w}{\partial \xi^4}\,.W d\xi d\eta = \frac{0.039365AD}{b^2 P^3} \tag{20}$$

$$\frac{2D}{P b^2}\int_0^1\int_0^1 \frac{\partial^4 w}{\partial \xi^2 \partial \eta^2}.\mathrm{W} d\xi d\eta = \frac{0.067075\mathrm{AD}}{\mathrm{b}^2 \mathrm{P}} \tag{21}$$

$$\frac{DP}{b^2}\int_0^1\int_0^1 \frac{\partial^4 w}{\partial \eta^4}.\mathrm{W} d\xi d\eta = \frac{0.007619AD}{b^2} P \tag{22}$$

$$\frac{1}{P}\int_0^1\int_0^1 \frac{\partial}{\partial \xi}\left(\frac{\partial w}{\partial \xi}\right).\mathrm{w} d\xi d\eta = \frac{0.000937A}{P} \tag{23}$$

Replacing (20), (21), (22) and (23) in (19) gives

$$N_x = \frac{\frac{0.039365\,AD}{b^2 P^3} + \frac{0.067075AD}{b^2 P} + \frac{0.007619\,AD}{b^2} P}{\frac{0.000937\,A}{P}} \tag{24}$$

$$N_x = \left(\frac{(0.039365 + 0.067075\,P^2 + 0.007619\,P^4)\frac{AD}{P^3 b^2}}{\frac{0.000937\,A}{P}}\right) \tag{25}$$

$$N_x = \left(\frac{(0.039365 + 0.067075\,P^2 + 0.007619\,P^4)\,D}{0.000937\,P^2 b^2}\right) \tag{26}$$

Introducing π^2 into (3.57) by multiplying both sides by π^2

$$\pi^2 N_x = \frac{D\pi^2}{b^2}\left(\frac{(0.039365 + 0.067075\,P^2 + 0.007619\,P^4)}{0.000937\,P^2}\right) \tag{27}$$

Taking π^2 *as* $\left(\frac{22}{7}\right)^2$, then

$$N_x = \frac{D\pi^2}{b^2}\left(\frac{(0.039365 + 0.067075\,P^2 + 0.007619\,P^4)7^2}{22^2\ X\ 0.000937 P^2}\right) \tag{28}$$

The critical load (force) is:

$$\boldsymbol{N}_{x(cr)} = \frac{\boldsymbol{D\pi^2}}{\boldsymbol{b^2}}\left(\frac{4.253255}{P^2} + 7.247226 + 0.823207P^2\right) \tag{29}$$

$$K = \left(\frac{4.253255}{P^2} + 7.247226 + 0.823207P^2\right) \tag{30}$$

K is the Buckling coefficient and coefficient of $\frac{D\pi^2}{b^2}$ which depend on the aspect ratio.

"D" means the modulus of rigidity of the plate and "b" means the edge of the plate that received the load.

RESULTS PRESENTATION AND DISCUSSIONS

The comparison of the data from this present study, Iyengar (1988) and Okpara (2014) solutions for buckling in the first mode is presented on table 1. Iyengar (1988) assumed trigonometric series in formulating the shape function to obtain buckling critical load

$$N_{x(cr)} = \left(\frac{4}{P^2} + 4P^2 + 2.667\right)\frac{D\pi^2}{b^2} \qquad 31$$

And Okpara (2014) studied the application of variational method in stability analysis of aircraft structures. Characteristic displacement polynomial functions were substituted in the indirect variational approach to stability analysis and the critical buckling loads for CCCC bounded plate amongst other panels considered in the study is stated below:

$$N_{x(cr)} = \frac{D\pi^2}{b^2}\left(\frac{4.2858}{P^2} + 2.4487 + 4.2858P^2\right) \qquad 32$$

Table 1.0: K – Values for different aspect ratios for CCCC bounded plate from previous studies and CFCF bounded plate from present study

Aspect Ratio Ä??Ä	K-values for CCCC plate. Iyengar (1988)	K-values for CCCC plate. Okpara (2014)	K-value for CFCF plate from PRESENT STUDY
0.1	402.707	431.072	432.5810
0.2	102.827	109.765	113.6115
0.3	47.471	50.454	54.5797
0.4	28.307	29.921	33.9618
0.5	19.667	20.663	24.4660
0.6	15.218	15.897	19.3582
0.7	12.790	13.295	16.3307
0.8	11.477	11.888	14.4198
0.9	10.845	11.211	13.1650
1.0	10.667	11.020	12.3237

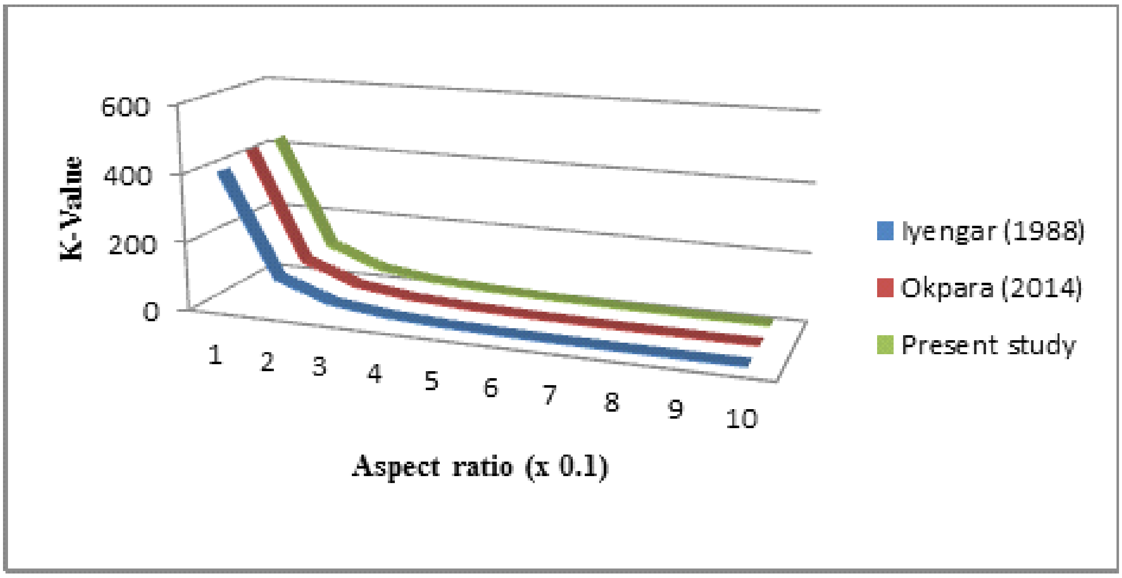

Figure 3: C**omparison for K-Values of** CCCC bounded plate from previous studies and CFCF bounded plate from present **(Correlation coefficient: Iyengar (1988) and present study = 0.9995; Okpara (2014) and present study = 0.9997)**

The K – Values for different aspect ratios increasing from 0.1 to 1.0 at 0.1 interval is shown on table 1.0 for rectangular plate with all edges clamped by Iyengar (1988), Okpara (2014) and CFCF bounded plate from present study. Figure 3.0 show values of CFCF bounded plate compared to CCCC bounded plate by iyengar (1988) and Okpara (2014). Correlation coefficient values of 0.9995 and 0.9997 are obtained when the values from present study are compared to solutions provided by Iyengar (1988) and Okpara (2014) respectively. These correlation coefficient values reflect a high degree of closeness in the values of this study and results from previous studies and it suggests the suitability of CFCF bounded plate in place of CCCC plate when stability is a factor of consideration in the analysis and design of structural plates for use in the elastic region. Also, the displacement functions formulated convergences rapidly and give very good approximation to the exact solution for application in stability analysis of clamped rectangular plates with opposite edges in the x-direction free.

CONCLUSIONS

Based on the results of this work, the following conclusions could be drawn:

1) The application of the buckling analysis solution (at aspect ratio 1.0) for rectangular plate with two opposite edges clamped and others edges free (K-value of 12.3237), offers a suitable replacement of rectangular plates with all boundaries clamped (K-values of 11.020 and 10.667).

2) This approach delivers a practicable approximate solution that save

cost, time and material in structural plate analysis and design

3) This research has presented an easy method rather than the complex trigonometric approach in finding solutions to rectangular plate with geometry of two opposite edges clamped and the other edges free as found in slab bridge decks and jetty platforms. This method can be adopted in their analysis when such structural plates are subjected to in-plane lateral loading.

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