Vol. 20, No. 1, 2023, ISSN 15965538, pp 56-71

Journal of Civil and Environmental Systems Engineering

Department of Civil Engineering, University of Benin, Nigeria

Journal homepage: https://j-cese.com/

Mathematical Application of Moment of Inertia of Rectangular Sections by Numerical Methods

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Abstract

Moment of inertia of cross-sectional properties of complex nature has been difficult to obtain analytically. Analysis computer programs, and calculus, have been applied in other to overcome the problems encountered in the calculation of this property efficiently and accurately. From literatures, the nine-noded Lagrangian quadratic element for an asymmetric section have been applied in creating shape functions at each nodes in two dimensions and have been incorporated into the double integration of the equations relating the shape functions obtained and the Jacobian determinant with respect to the reference z-coordinates numerically and utilized in deriving the moment of inertia of a solid cross-section. In this paper, this equation has been applied to derive the moment of inertia for a four-noded and nine-noded symmetrical rectangular beam cross section numerically. Numerically, there was no difference between the moment of inertia calculated from the four-noded and nine-noded rectangular elements. The results were not different from the analytical value obtained from literatures which showed a very good agreement.

Keywords: Rectangular element, Lagrange function, shape function, Jacobian matrix, Moment of inertia

Introduction

Reference cross sections, according to Pilkey (2002), can be utilized to ease computations when dealing with complex form cross sections. A transformation maps the geometry of a reference cross section with its simple geometry to the geometry of the real cross section. This transformation describes the coordinates of each real domain point (y, z) in terms of the coordinates of the associated reference domain point (η, ζ). Isoparametric elements have geometric transformation equations and function interpolation formulas that have the same form. In terms of shape functions, the geometrical transformation for the nine-noded Lagrangian element is defined as:

$$y(\eta, \zeta) = N(\eta, \zeta)y$$

$$z(\eta, \zeta) = N(\eta, \zeta)z$$

Where:

N is a row vector of length 9 whose entries are the shape functions

y, z are the y, z coordinates of the nodes of the real element and

η, ζ are the η, ζ coordinates of the nodes of the reference element.

Assume that at any location within the element, the basic variable is a function of values at the element's nodal points. Shape function refers to the function that connects the field variable at any location within the element to the field variables of nodal points. This is also known as the interpolation function or the approximation function. (Bharikatti, 2005).

There are certain fundamental unknowns in engineering problems. If they are discovered, the overall behavior of the structure can be anticipated. These unknowns are endless in a continuum. By splitting the solution region into small portions called elements and describing the unknown field variables in terms of presumed approximation functions (Interpolating functions/Shape functions) within each element, the finite element approach reduces such unknowns to a finite number. The approximation functions are described in terms of field variables of nodes or nodal points. Thus, the unknowns in finite element analysis are the field variables of the nodal points. Once these have been determined, the field variables at any time can be determined using interpolation functions/Shape functions (Reddaiah, 2017).

When working with elements of complex shapes, reference elements can be utilized to ease calculations. A transformation maps the geometry of a reference element with its simple geometry to the geometry of the real element. The coordinates of each point in the real domain are defined in terms of the coordinates of the corresponding point in the reference domain via this transformation (Pilkey, 2002).

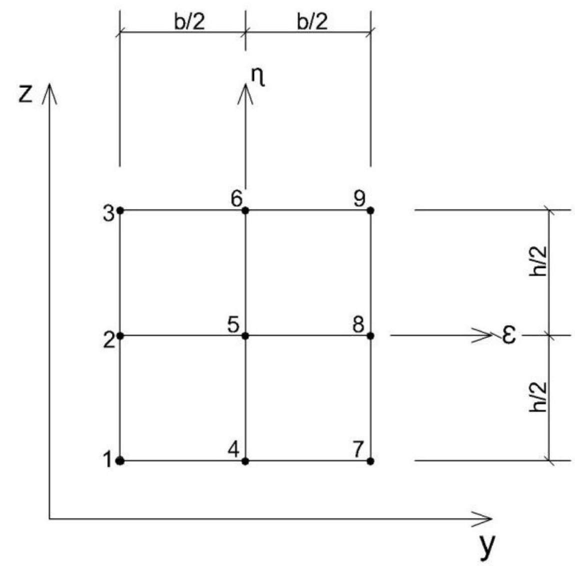

Figure 1. The transformed nine noded rectangular section.

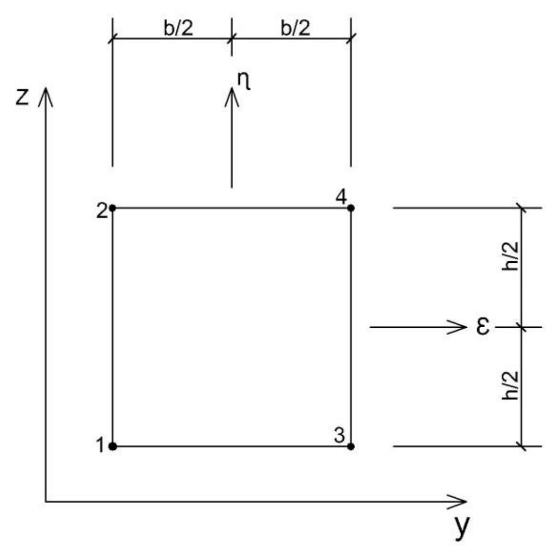

Figure 2. The transformed four noded rectangular section.

Reddaiah (2017) used Lagrange Functions in Natural Coordinate System to generate Shape Functions for 9-Noded Rectangular Elements and proved two shape function verification requirements. The first verification condition is that the sum of all the shape functions equals one, and the second is that each shape function has a value of one at its own node and zero at the other nodes.

Sangare et al. (2009) conducted a comparative study of the hierarchical p-element shape functions: noninterference condition formulation, Lagrange formulation, and Legendre formulation for both quadrilateral and triangular elements using a set of linear elastic two-dimensional numerical applications. For preliminary comparisons, the meshes were primarily composed of 9 node quadrilateral and 7 node triangular elements. These analyses show that, while the Legendre type formulation has a superior condition number of stiffness matrix, it is not the optimum p-element formulation for deformed meshes or for convergence stability of computed stress values.

EUREQA was utilized to find analytical models for adjusting an existing equation used to calculate the improvement of the moment of inertia on a ribbed portion in reference to the same item. The analytical and numerical answers obtained using the classical Orlov's equation differ slightly. The goal of the project was to create an analytical model based on Orlov's to increase the accuracy of the results. (Hugo and Jerzy, 2020).

Shear stress-related cross-sectional characteristics have historically been difficult to acquire analytically. The torsional constant, shear deformation coefficients, warping constant, and shear stresses are all examples of these properties. Pilkey, (2002) outlined the fundamental concepts of finite element analysis utilized in the analysis of asymmetric cross-sections and showed how warping-independent section parameters, such as areas or moments of inertia, are determined through integration. The finite element method was described in terms of a nine-node Lagrangian element in two dimensions.

Patela et al. (2005) developed an explicit equation for the effective moment of inertia taking cracking into account for uniformly distributed loaded reinforced concrete (RC) beams in order to estimate short-term deflection at service load. The trained neural network produced the explicit expression. Three major structural parameters were chosen as neural network inputs. Three major structural parameters were chosen as neural network inputs. The neural network training data sets were prepared using the finite element software ABAQUS. A sensitivity analysis was undertaken, and the results show that the effective moment of inertia is highly dependent on the input parameters.

Various processes and empirical formulas were used to undertake manual analysis and design difficulties.Many times, redesigning the section is required to satisfy the codal provisions, which takes extra time and energy. A programming language was applied to calculate the Moment of Inertia of various geometrical shapes such as a circle, rectangle, triangle, and the unsymmetrical I section, channel section, T section, and L section using the basic concept of C programming and condition of If statements. This method's result was compared to the appropriate analytical procedure. Rani,(2021)

Ochiai,(2019) established boundary integral equations to calculate the moment of inertia of a 3D nonhomogeneous material. A boundary element approach formulization was used, and a strategy for direct numerical integration of the three-dimensional domain utilizing a three-dimensional interpolation method without domain division was presented. To test the numerical integral's accuracy, the moment of inertia of the spherical domain of radius R was measured and compared to the Monte Carlo approach.

The mass moment of inertia (MMI) of a wooden beam with circular, circular hollow, rectangular, and rectangular hollow cross sections was computed and plotted using MAPLE to determine whether there is a direct or indirect relationship between the MMI and AMI. Aside from the fact that the results were consistent with the literature, the MMI of all the beams studied was more than their AMI. The AMI for the beams along another axis was also greater than the solid's moment of inertia about the axis through the solid's center of mass, given the shortest distance between the axes. Agarana et al. (2021).

As an alternative to the parallel axis theorem, calculus and software applications, this paper aims at deriving the moment of inertia for a symmetric rectangular beam cross section by applying the double integral equation of Pilkey, (2002) which related shape functions, N, with the z-coordinates of each nodal points and the Jacobean determinant, |J| with respect to the the non-dimensional axis ξ and η, from first principles numerically.

Methodology

THE SHAPE FUNCTION FOR THE 9-NODED QUADRATIC RECTANGULAR SECTION

In terms of the shape function, Chandrupatla and Belegundu, (2002) defined the geometrical transformation for the nine-node Lagrangian element as:

$$N_1 = \frac{\eta\xi(1-\xi)(1-\eta)}{4}$$

$$N_2 = \frac{\eta\xi(1+\xi)(1-\eta)}{4}$$

$$N_3 = \frac{\eta\xi(1+\xi)(1+\eta)}{4}$$

$$N_4 = \frac{-\eta\xi(1-\xi)(1+\eta)}{4}$$

$$N_5 = \frac{-\eta(1-\eta)(1-\xi)(1+\xi)}{2}$$

$$N_6 = \frac{\xi(1-\eta)(1+\eta)(1+\xi)}{2}$$

$$N_7 = \frac{\eta(1+\eta)(1-\xi)(1+\xi)}{2}$$

$$N_8 = \frac{-\xi(1-\eta)(1+\eta)(1-\xi)}{2}$$

$$N_9 = (1-\xi)^2(1-\eta)^2 \qquad (1)$$

TRANSFORMATIONS OF DERIVATIVES AND INTEGRALS

The Jacobian matrix relates partial derivatives of a function f((η, ξ) with respect to η and ξ in the reference domain , to its derivatives with respect to the y and z axis in the real domain.
For a nine-noded quadratic rectangle lagrangian element, the Jacobian matrix, J is given by Pilky, 2002:

$$\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & \frac{\partial N_5}{\partial \xi} & \frac{\partial N_6}{\partial \xi} & \frac{\partial N_7}{\partial \xi} & \frac{\partial N_8}{\partial \xi} & \frac{\partial N_9}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & \frac{\partial N_5}{\partial \eta} & \frac{\partial N_6}{\partial \eta} & \frac{\partial N_7}{\partial \eta} & \frac{\partial N_8}{\partial \eta} & \frac{\partial N_9}{\partial \eta} \end{bmatrix} \begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \\ y_3 & z_3 \\ y_4 & z_4 \\ y_5 & z_5 \\ y_6 & z_6 \\ y_7 & z_7 \\ y_8 & z_8 \\ y_9 & z_9 \end{bmatrix} \qquad (2)$$

Differentiating equation 1,with respect to η ,we obtain the partial derivatives of the shape functions as:

$$\frac{\partial N_1}{\partial \eta} = \frac{\xi(1-2\eta)(1-\xi)}{4}$$

$$\frac{\partial N_2}{\partial \eta} = \frac{-(1-2\eta)(1-\xi)}{2}$$

$$\frac{\partial N_3}{\partial \eta} = \frac{-\xi(1-2\eta)(1+\xi)}{4}$$

$$\frac{\partial N_4}{\partial \eta} = 4\eta\xi(1-\xi)$$

$$\frac{\partial N_5}{\partial \eta} = -8\eta(1-\xi^2)$$

$$\frac{\partial N_6}{\partial \eta} = -4\eta\xi(1+\xi)$$

$$\frac{\partial N_7}{\partial \eta} = -\xi(1+2\eta)(1-\xi)$$

$$\frac{\partial N_8}{\partial \eta} = 2(1+2\eta)(1-\xi^2)$$

$$\frac{\partial N_9}{\partial \eta} = \xi(1+2\eta)(1+\xi) \qquad (3)$$

Similarly,

$$\frac{\partial N_1}{\partial \xi} = \frac{\eta(1-\eta)(1-2\xi)}{4}$$

$$\frac{\partial N_2}{\partial \xi} = \eta\xi(1-\eta)$$

$$\frac{\partial N_3}{\partial \xi} = \frac{-\eta(1-\eta)(1+2\xi)}{4}$$

$$\frac{\partial N_4}{\partial \xi} = \frac{-(1-\eta^2)(1-2\xi)}{2}$$

$$\frac{\partial N_5}{\partial \xi} = -2\xi(1-\eta^2)$$

$$\frac{\partial N_6}{\partial \xi} = \frac{(1-\eta^2)(1+2\xi)}{2}$$

$$\frac{\partial N_7}{\partial \xi} = \frac{-\eta(1+\eta)(1-2\xi)}{4}$$

$$\frac{\partial N_8}{\partial \xi} = -\eta\xi(1+\eta)$$

$$\frac{\partial N_9}{\partial \xi} = \frac{\eta(1+\eta)(1+2\xi)}{4} \quad (4)$$

The determinant of the jacobian, J, becomes

$$| J | = J_{11} J_{22} - J_{12} J_{21} \quad (5)$$

The determinant | J | can be expressed in terms of the shape function derivatives and the element nodal coordinate vectors as: Pilky, (2002).

$$| J | = \frac{\partial N}{\partial \eta} y \frac{\partial N}{\partial \xi} z - \frac{\partial N}{\partial \eta} z \frac{\partial N}{\partial \xi} y \quad (6)$$

From figure 1, the y-coordinates are,

$$y_1 = \frac{-b}{2}$$
$$y_2 = \frac{-b}{2}$$
$$y_3 = \frac{-b}{2}$$
$$y_4 = 0$$
$$y_5 = 0$$
$$y_6 = 0$$
$$y_7 = \frac{b}{2}$$
$$y_8 = \frac{b}{2}$$
$$y_9 = \frac{b}{2} \quad (7)$$

and the z-coordinates are,

$$z_1 = \frac{-h}{2}$$
$$z_2 = 0$$
$$z_3 = \frac{h}{2}$$
$$z_4 = \frac{-h}{2}$$
$$z_5 = 0$$
$$z_6 = \frac{h}{2}$$
$$z_7 = \frac{-h}{2}$$
$$z_8 = 0$$
$$z_9 = \frac{h}{2} \quad (8)$$

Substituting equation 3 and equation 4 into equation 6, and putting ξ=0, gives the values for J_{11}, J_{12}, J_{21}, and J_{22}:

$J_{11} = \frac{\partial N}{\partial \eta} y =$

$[\frac{\xi(1-2\eta)(1-\xi)}{4} x \frac{-b}{2} + [\frac{-2(1-2\eta)(1-\xi^2)}{4} x \frac{-b}{2}]$

$+[\frac{-\xi(1-2\eta)(1+\xi)}{4} x \frac{-b}{2}] + \frac{-4\eta\xi(1-\xi)x0}{4} + [\frac{-8\eta(1-\xi^2)x0}{4}] + [\frac{-4\eta\xi(1+\xi)x0}{4}] + [\frac{-\xi(1+2\eta)(1-\xi)}{4} x \frac{b}{2}] + [\frac{2(1+2\eta)(1-\xi^2)}{4} x \frac{b}{2}$
$] + [\frac{\xi(1+2\eta)(1-\xi^2)}{4} x0]$

But = ξ=0

$J_{11} = \frac{-2(1-2\eta)}{4} x \frac{-b}{2}] + \frac{2(1+2\eta)}{4} x \frac{b}{2}$

$= (\frac{-2}{4} + \frac{4\eta}{4}) x \frac{-b}{2} + (\frac{2}{4} + \frac{4\eta}{4}) x \frac{b}{2}$

$= \frac{b}{2} - \frac{b\eta}{2} + \frac{b}{4} + \frac{b\eta}{2}$

But = ξ=0

$J_{11} = \frac{b}{2}$ (9)

$J_{12} = \frac{\partial N}{\partial \eta} z$

$J_{12} = [\frac{\xi(1-2\eta)(1-\xi)}{4} x \frac{-h}{2} + [\frac{-2(1-2\eta)(1-\xi^2)}{4} x0]$
$+[\frac{-\xi(1-2\eta)(1+\xi)}{4} x \frac{h}{2}] + \frac{-4\eta\xi(1-\xi)}{4} x \frac{-h}{2}$
$+[\frac{-8\eta(1-\xi^2)x0}{4}] + [\frac{-4\eta\xi(1+\xi)}{4} x \frac{h}{2}]$
$+[\frac{-\xi(1+2\eta)(1-\xi)}{4} x \frac{-h}{2}] + [\frac{2(1+2\eta)(1-\xi^2)}{4} x0$
$+ [\frac{\xi(1+2\eta)(1-\xi^2)}{4} x0]$

But = ξ=0

$J_{12} = 0$ (10)

$J_{21} = \frac{\partial N}{\partial \xi} y$

$= [\frac{\eta(1-\eta)(1-2\xi)}{4} x \frac{-b}{2} + [\frac{-4(1-\eta)}{4} x \frac{-b}{2}]$

$+[\frac{-\eta(1-\eta)(1+2\zeta)}{4} x \frac{-b}{2}] + \frac{-2\xi(1-\eta^2)x(1-2\xi)x0}{4} + [\frac{-8\xi(1-\eta^2)x0}{4}] + [\frac{2(1-\eta^2)x(1+2\xi)x0}{4}] + [\frac{-\eta(1+\eta)(1-2\xi)}{4} x \frac{b}{2}] +$
$[\frac{-4\eta\zeta(1+\eta)}{4} x \frac{b}{2}] + [\frac{\eta(1+\eta)(1+2\xi)}{4} x \frac{b}{2}]$

But = η=0

$$J_{21} = 0 \quad (11)$$

$$J_{22} = \frac{\partial N}{\partial \xi} z$$

Similarly,

$$J_{22} \ \frac{-2(1-2\xi)}{4} x \frac{-h}{2}] + \frac{2(1+2\xi)}{4} x \frac{h}{2}$$

$$= (\frac{-2}{4} + \xi) \ x \ \frac{-h}{2} + (\frac{2}{4} + \xi) \ x \ \frac{h}{2}$$

$$= \frac{h}{4} - \frac{h\xi}{2} + \frac{h}{4} + \frac{h\xi}{2}$$

$$J_{22} = \frac{h}{2} \quad (12)$$

The Jacobian matrix becomes:

$$J = \begin{bmatrix} \frac{b}{2} & 0 \\ 0 & \frac{h}{2} \end{bmatrix} \quad (13)$$

The Jacobian Determinant becomes:

$$|J| = \frac{bh}{4} \quad (14)$$

MOMENT OF INERTIA OF THE NINE -NODED RECTANGULAR SECTION, $I_{yR.}$

The moment of inertia, $I_{yR.}$, can be solved using the following equations by (Pilkey, 2002):

$$I_{yR} = \sum_{e=1}^{2} \int_{-1}^{1} \int_{-1}^{1} (Nz_e)^2 \ |J_e| d\eta \ d\xi \quad (15)$$

$I_{yR} =$

$$\sum_{e=1}^{2} \int_{-1}^{1} \int_{-1}^{1} \left([N_1 \ \ N_2 \ \ N_3 \ \ N_4 \ \ N_5 \ \ N_6 \ \ N_7 \ \ N_8 \ \ N_9 \ ] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \end{bmatrix} \right)^2 |J| \ d\eta \ d\xi \quad (16)$$

Substituting the shape functions and the nodal coordinates into equation 15 gives the moment of inertia, as:

$$I_{yR} = \sum_{e=1}^{2} \int_{-1}^{1}\int_{-1}^{1} [(\frac{4\eta\xi(1-\eta)(1-\xi)}{4} x \frac{h}{2}) + (\frac{-2\eta(1-\eta)(1-\xi^2)x0}{4}) + \left(\frac{-\eta\xi(1-\eta)(1+\xi)}{4} x \frac{h}{2}\right) + (\frac{-2\xi(1-\eta^2)(1-\xi)}{4} x \frac{-h}{2}) + (\frac{-4(1-\eta^2)(1-\xi^2)x0}{4} + \left(\frac{2\xi(1-\eta^2)(1+\xi)}{4} x \frac{h}{2}\right) + \left(\frac{-\eta\xi(1+\eta)(1-\xi)}{4} x \frac{-h}{2}\right) + (\frac{2\eta(1+\eta)(1-\xi^2)x0}{4} + \left(\frac{\eta\xi(1+\eta)(1+\xi)}{4} x \frac{h}{2}\right) |J_{eR}| d\eta\, d\xi$$

$$= \sum_{e=1}^{2} \int_{-1}^{1}\int_{-1}^{1} [\left(\frac{-h\eta\xi(1-\xi-\eta+\eta\xi)}{8}\right) - \left(\frac{h\eta\xi(1+\xi-\eta-\eta\xi)}{8}\right)$$

$$+ \left(\frac{h\xi(1-\eta^2-\eta-\eta\xi)}{4}\right) + \left(\frac{h\xi(1+\xi-\eta^2-\eta^2\xi)}{4}\right) + \left(\frac{h\eta\xi(1-\xi+\eta-\eta\xi)}{8}\right) + \left(\frac{h\eta\xi(1+\xi+\eta+\eta\xi)}{8}\right)]^2 x \frac{bh}{4} d\eta d\xi$$

$$= \sum_{e=1}^{2} \int_{-1}^{1}\int_{-1}^{1} [\left(\frac{h\xi}{2}\right)]^2 x \frac{bh}{4}\, d\eta\, d\xi$$

$$= \sum_{e=1}^{2} \int_{-1}^{1}\int_{-1}^{1} [\left(\frac{h^2\xi^2}{4}\right) x \frac{bh}{4}] d\eta\, d\xi$$

$$= \sum_{e=1}^{2} \int_{-1}^{1}\int_{-1}^{1} \left(\frac{bh^3\xi^2}{16}\right) d\eta d\xi$$

$$= \int_{-1}^{1} \left(\frac{bh^3\xi^2}{16}\right)_{-1}^{1} d\eta$$

$$= \int_{-1}^{1} \left(\frac{bh^3}{16} x \frac{1}{3}\right) - \left(\frac{bh^3}{16} x \frac{-1}{3}\right) d\eta$$

$$= \int_{-1}^{1} \left(\frac{bh^3}{48}\right) + \left(\frac{bh^3}{48}\right) d\eta$$

$$= \int_{-1}^{1} \left(\frac{bh^3}{24}\right) d\eta$$

$$I_{yR} = \frac{bh^3}{12} \tag{17}$$

THE SHAPE FUNCTION FOR THE 4-NODED QUADRATIC RECTANGLE ELEMENT

The shape function for a four noded rectangular elemrnt is given by: Bharikatti,(2005).

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}$$
$$N_2 = \frac{(1-\xi)(1+\eta)}{4}$$

$$N_3 = \frac{(1+\xi)(1-\eta)}{4}$$
$$N_4 = \frac{(1+\xi)(1+\eta)}{4} \qquad (18)$$

Differentiating with respect to η gives:

$$\frac{\partial N_1}{\partial \eta} = \frac{(-1+\xi)}{4}$$

$$\frac{\partial N_2}{\partial \eta} = \frac{(1-\xi)}{4}$$

$$\frac{\partial N_3}{\partial \eta} = \frac{(-1-\xi)}{4}$$

$$\frac{\partial N_4}{\partial \eta} = \frac{(1+\xi)}{4} \qquad (19)$$

Similarly,

$$\frac{\partial N_1}{\partial \xi} = \frac{(-1+\eta)}{4}$$

$$\frac{\partial N_2}{\partial \xi} = \frac{(-1-\eta)}{4}$$

$$\frac{\partial N_3}{\partial \xi} = \frac{(1-\eta)}{4}$$

$$\frac{\partial N_4}{\partial \xi} = \frac{(1+\eta)}{4} \qquad (20)$$

From the geometry in figure 2, the coordinates of the four noded points are given as:

$y_1=\frac{-b}{2}$
$y_2=\frac{-b}{2}$
$y_3=\frac{b}{2}$
$y_4=\frac{b}{2}$
$z_1=\frac{-h}{2}$
$z_2=\frac{h}{2}$
$z_3=\frac{-h}{2}$
$z_4=\frac{h}{2}$ (21)

Similarly, substituting equation 19 and equation 20 into equation 6, and putting ξ=0, gives the values for J_{11}, J_{12}, J_{21}, and J_{22} for the 4 noded section:

$$J_{11}=\frac{\partial N}{\partial \eta}y = [\frac{(-1+\xi)}{4}.\frac{-b}{2}] +[\frac{(1-\xi)}{4}.\frac{-b}{2}]+[\frac{(-1-\xi)}{4}.\frac{b}{2}]+[\frac{(1+\xi)}{4}.\frac{b}{2}]$$

$$J_{11}=0 \quad (22)$$

$$J_{12} = \frac{\partial N}{\partial \eta}z = [\frac{(-1+\xi)}{4}.\frac{-h}{2}+[\frac{(1-\xi)}{4}.\frac{h}{2}]+ \quad [\frac{(-1-\xi)}{4}.\frac{-h}{2}]+[\frac{(1+\xi)}{4}.\frac{h}{2}]$$

$$J_{12}=\frac{h}{2} \quad (23)$$

$$J_{21}=\frac{\partial N}{\partial \xi}y = [\frac{(-1+\eta)}{4}.\frac{-b}{2}+[\frac{(-1-\eta)}{4}.\frac{-b}{2}]+ [\frac{(1-\eta)}{4}.\frac{b}{2}]+[\frac{(1+\eta)}{4}.\frac{b}{2}]$$

$$J_{21}=\frac{b}{2} \quad (24)$$

$$J_{22}=\frac{\partial N}{\partial \xi}z=[\frac{(-1+\eta)}{4}.\frac{-h}{2}+[\frac{(-1-\eta)}{4}.\frac{h}{2}]+[\frac{(1-\eta)}{4}.\frac{-h}{2}]+[\frac{(1+\eta)}{4}.\frac{h}{2}]$$

$$J_{22} = 0 \tag{25}$$

The Jacobian matrix becomes:

$$J = \begin{bmatrix} 0 & \frac{b}{2} \\ \frac{h}{2} & 0 \end{bmatrix} \tag{26}$$

The Jacobian Determinant becomes:

$$|J| = \frac{bh}{4} \tag{27}$$

MOMENT OF INERTIA OF THE FOUR -NODED RECTANGULAR SECTION,

The moment of inertia, , can be solved using equation 15. For the 4 noded section, the moment of inertia becomes:

$$\sum_{e=1}^{2} \int_{-1}^{1} \int_{-1}^{1} \left(\{N_1 \quad N_2 \quad N_3 \quad N_4\} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} \right)^2 |J| d\eta d\xi \tag{28}$$

Substituting equation 18 and equation 21 into equation 28 gives:

$$= \sum_{e=1}^{2} \int_{-1}^{1} \int_{-1}^{1} \left(\frac{(h\eta)}{2}\right)^2 d\eta d\xi$$

$$= \int_{-1}^{1} \left(\frac{bh^3\eta^2}{16}\right) |_{-1}^{1} \, d\xi$$

$$= \int_{-1}^{1} \left(\frac{bh^3}{16} \text{x} \frac{1}{3}\right) - \left(\frac{bh^3}{16} \text{x} \frac{-1}{3}\right) d\xi$$

$$= \int_{-1}^{1} \left(\frac{bh^3}{48}\right) + \left(\frac{bh^3}{48}\right) d\xi$$

$$= \int_{-1}^{1} \left(\frac{bh^3}{24}\right) d\xi$$

$$I_{yR} = \frac{bh^3}{12} \qquad (29)$$

The analytical value of the moment of inertia of a rectangular section is given by Rhyder, (1982.) as:

$$I = \frac{bh^3}{12} \qquad (30)$$

Discussion

The moment of inertia for a rectangular section was calculated numerically for the nine noded lagrangian quadratic element and the four noded lagrangian quadratic element. The cross section was meshed into a four-noded and nine-noded element for a symmetrical rectangular solid cross section, and the equation of the moment of inertia of the rectangular sections were obtained by double integration of the equations relating shape functions obtained from literature, with the transformed coordinate points from the cross section, and the Jacobian determinant as obtained by Pilkey, (2002).

It was discovered that the shape functions for the nodal points were related to the moment of inertia of the cross section, which was also confirmed by literature. The values of the moment of inertia are observed to be the same for both the 4 noded cross section and the 9 noded section, indicating that the number of nodal points had no effect on the moment of inertia. This is because the moment of inertia is a cross sectional property that does not vary regardless of the number of nodal points unless the sectional dimensions are changed. The moment of inertia calculated was with respect to the centroidal axis. The moment of inertia was the same for both the analytical solution obtained from literature and the numerical method.

Conclusion and Recommendation

The numerically determined moment of inertia equation of this work provides a valid solution for estimating the moment of inertia of rectangular cross-sections.

In the case of a symmetric rectangular plain section, increasing the number of crossection node points from a four noded rectangular section to a nine noded rectangular section has no effect on the moment of inertia.

This method is applicable to other complex symmetrical and asymmetrical shapes. More work may be done to investigate the effect of mesh size and number on the moment of inertia of rectangular beams using numerical methods for mesh sizes larger than nine nodal points.

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